CHAPTER-2

LAPLACE EQUATION AND ITS SOLUTION

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2.1 Introduction

To model the physical problems, the partial differential equations (PDEs) are the common method**.** PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, gravitation and quantum mechanics, etc. In this chapter, we will discuss about different types of the partial differential equations, their classifications and the classical and weak solutions, etc.

Partial Differential Equation

A partial differential equation (PDE) is differential equation that contain an unknown function and its partial derivate with respect to two or more variables i.e., let U be an open subset of R^n. An expression of the form

$$F(D^k u(x), D^{k-1} u(x), ..., Du(x), u(x), x) = 0 \quad (x \in U) \qquad \ldots(1)$$

is called a kth-order partial differential equation, where

$F : R^{n^k} \times R^{n^{k-1}} \times ... \times R^n \times R \times U \to R$ is given and $u : U \to R$ is the unknown.

Example: The equation $u_t + u_x = 0$ is a partial differential equation, the unknown function is u and independent variables are x and t.

2.1.1 Classifications of Partial Differential Equations

Partial Differential Equations can be classified into four types

(a) Linear (b) Semi-linear (c) Quasi-linear (d) Non-linear.

(a) **Linear Partial Differential Equation**: A Partial Differential Equation (1) is said to linear PDE if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x) \qquad \dots (2)$$

for a given function $a_\alpha\left(\alpha \le k\right)$ and f Here, it is clear that the coefficients of derivate are a function of x only. The above equation is said to be homogeneous if *f=0*.

For example: $u_t + u_x = 0$ is a transport equation which is of first order, linear and homogeneous.

Some famous linear PDE are

1. Laplace equation $\Delta u = 0$ or $\sum_i u_{xx} = 0$
2. Linear Transport Equation $u_t + \bar{b}\Delta u = 0,\ \bar{b} \in R^n$
   $Du = \left(u_{x_1}, u_{x_2} \ldots u_{x_n}\right)$
3. Heat (Diffusion) Equation $u_t - \Delta u = 0$
4. Wave equation $u_{tt} - \Delta u = 0$

(b) **Semi-linear Partial Differential Equation**: A Partial Differential Equation (1) is said to semi-linear PDE if it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0\left(D^{\alpha-1}u, \ldots, Du, u, x\right) = 0, \qquad \dots (3)$$

Here, coefficient of highest order derivative is a function of x only.

For example: $a(x)u_{xx} + u_x u_t = 0$.

(c) Quasi-linear Partial Differential Equation: A Partial Differential Equation (1) is said to quasi PDE if it has the form

$$\sum_{|\alpha| = k} a_\alpha(D^{\alpha-1}u, \ldots, Du, u, x) D^\alpha u + a_0\left(D^{\alpha-1}u, \ldots, Du, u, x\right) = 0, \qquad \dots (4)$$

Here, coefficient of highest order derivative are lower order derivative and function of *x* but not same order derivatives.

For example: $u_x u_{xx} + u_x u_t = 0$

(d) Nonlinear Partial Differential Equation: A Partial Differential Equation is non-linear in the highest order derivatives.

For example: $u^2 u_{xx} + u_x u_t = 0$

2.1.2 System of Partial Differential Equations

An expression of the form is said to be **system of partial differential equations** if it is represented by

$$\overline{F}(D^k\overline{u}(x), D^{k-1}\overline{u}(x), ...D\overline{u}(x), \overline{u}(x), x) = 0 \quad (x \in U)$$

is called a *k*th order system of partial differential equations in *u* where

$$\overline{F}: R^{mn^k} \times R^{mn^{k-1}} \times ... \times R^{mn} \times R^m \times U \to R^m$$

is given and $\overline{u} = \left(u^1, u^2, ..., u^m\right)$ is the unknown function such that $\overline{u}: U \to R^m$

For example:

$$\mu\Delta\overline{u} + (\lambda + \mu)\, div\overline{u} = 0 \quad \text{where } \overline{u} = \left(u^1, u^2, u^3\right)$$

Note: The classifications of system of partial differential equations are same as in case of a partial differential equations**.**

2.1.3 Solution of PDE

An expression u which satisfies the given PDE (1) is called a solution of the Partial Differential Equation.

Well posed problem: A given problem in Partial Differential Equation is well posed (Hadaward) if it satisfies

(i) existence
(ii) uniqueness
(iii) continuously depend on the data of given problem.

Classical Solution: If a solution of a given problem satisfies the above three conditions i.e., the solution of k^{th} order partial differential equation exists, is unique and is at least k times differentiable, then the solution is called classical solution. Solutions of Wave equation, Lalpace, and Heat equation etc., are classical solutions.

Weak Solution: If a solution of a given problem exists and is unique but does not satisfy the conditions of differentiability, such solution is called weak solution.

For Example: The gas conservation equation

$$u_t + F(u_x) = 0$$

models a shock wave in particular situation. So solutions exists, is unique, but not continuous. Such solution is known as weak solution.

Note: There are several physical phenomenon in which the problem has a unique solution, but does not satisfy the condition of differentiability. In such cases, we cannot claim that we are not able to find the solution rather such solutions are called weak solutions

2.2 Transport Equation

Homogeneous Transport Equation

The simplest partial differential equation out of four important equations is the Transport equation with constant coefficient

$$u_t + b\,.\,Du = 0 \qquad \ldots (1)$$

in $R^n \times (0,\infty)$, where $b = (b_1, b_2, b_3 ..., b_n)$ is a fixed vector in R^n and $u : R^n \times [0,\infty) \to R$ is the unknown function $u = u(x, t)$. Here $x = (x_1, ..., x_n) \in R^n$ denotes a typical point in space and $t \geq 0$ is the time variable.

Theorem: Initial Value Problem

Consider the homogeneous transport equation

$$u_t + b\,.\,Du = 0 \quad \text{in} \quad R^n \times [0,\infty) \qquad \ldots(1)$$

$$u = g \quad on \quad R^n \times \{t = 0\} \qquad \ldots(2)$$

where $b \in R^n$ and $g : R^n \to R$ is known and $Du = D_x u = \left(u_{x_1}, ..., u_{x_n}\right)$ for the gradient of u with respect to the spatial variables x. The problem is to compute $u(x,t)$.

Solution:

Let (x,t) be any point in the $R^n \times [0,\infty)$. To solve equation (1) , we observe the L.H.S. of equation (1) carefully, we find that it denotes the dot product of $\left(u_{x_1}, ..., u_{x_n}, u_1\right)$ with $(b_1, ..., b_n, 1)$. So L.H.S. of equation (1) tells that the derivative of u in the direction of $(b,1)$ is zero in R^{n+1} dimensional space. So, the line through (x,t) in the direction of $(b,1)$ is

$$\left.\begin{aligned} x(s) &= x + sb \\ t(s) &= t + s \end{aligned}\right\}, \quad s \in R \qquad \ldots (3)$$

This line hits the plane $\Gamma := R^n \times \{t = 0\}$ at the point $(x - tb, 0)$ when $s = -t$.

Define a parametric equation of line in the direction $(b,1)$ is

$$z(s) = u(x + sb, t + s) \qquad \ldots (4)$$

where $s \in R$ is the parameter and $z : R \to R$.

Then, differentiating (4) w.r.t. s, we get

$$\begin{aligned} \dot{z}(s) &= Du(x + sb, t + s).b + u_t(x + sb, t + s) \\ &= 0 \end{aligned} \qquad \text{(using (1))}$$

$\Rightarrow z(s)$ is a constant function of s on the line (3).

$\Rightarrow$ u is constant on the line (4) through (x,t) with the direction $(b,1) \in R^{n+1}$.

and $u(x-tb,0) = g(x-tb)$

By virtue of given initial condition (2), we deduce that

$$u(x,t) = g(x-tb) \qquad \dots (5) \text{ for } x \in R^n \text{ and } t \geq 0.$$

Hence, if we know the value of u at any point on each such line, we know its value everywhere in $R^n \times (0,\infty)$ and it is given by equation (5).

Conversely, if $g \in C^1$, then $u = u(x,t)$ defined by (5) is indeed a solution of given initial value problem.

From (5), we find that

$$u_t = -b.D(x-tb)$$

and

$$Du=Dg$$

Hence $u_t + b.Du = -b.Dg + b.Dg = 0$ for (x ,t) in $R^n \times [0,\infty)$ and for t=0 $u(x,0) = g(x)$ on $R^n \times \{t=0\}$

Remark: If g is not C^1, then there is obviously no C^1 solution of (1). But even in this case formula (5) certainly provides a strong and in fact the only reasonable, candidate for a solution. We may thus informally declare $u(x,t) = g(x-tb)\left(x \in R^n, t \geq 0\right)$ to be a weak solution of given initial value problem even should g not be C^1. This all makes sense even if g and thus u are discontinuous. Such a notion, that a non-smooth or even discontinuous function may sometimes solve a PDE will come up again later when we study nonlinear transport phenomenon.

2.3 Non-homogenous Problem

Theorem: Consider the non–homogeneous initial value problem of transport equation

$$u_t + b.Du = f(x,t) \quad in \quad R^n \times (0,\infty) \qquad \dots(1)$$

$$u = g \qquad \text{on } R^n \times \{t=0\} \qquad \dots(2)$$

where $b \in R^n$, $g : R^n \to R$, $\mathrm{f} : \mathrm{R}^n \times [0,\infty) \to R$ is known and $Du = D_x u = \left(u_{x_1}, \dots, u_{x_n}\right)$ for the gradient of u with respect to the spatial variables x. Solve the equation for u=u(x,t) with initial condition (2).

Solution: Fix a point $(x,t) \in R^{n+1}$, as discussed before, the equation of line passing through (x,t) in the direction of $(b,1)$ is given by $z(s) = u(x+sb, t+s)$, where s is the parameter.

Differentiating this w. r. t. s

$$\dot{z}(s) = f(x+sb, t+s) \qquad \text{(using (1))}$$

Integrating w. r. t. s from –t to 0

$$\int_{-t}^{0} \dot{z}(s)ds = \int_{-t}^{0} f(x+sb, t+s)ds$$

$$z(0)-z(-t) = \int_{-t}^{0} f(x+sb, t+s)ds$$

Substitute t+s= ψ , ds=d ψ

$$z(0)-z(-t) = \int_{0}^{t} f(x+b(\psi-t), \psi)d\psi$$

$$u(x,t)-u(x-bt,0) = \int_{0}^{t} f(x+b(s-t), s)ds \qquad (\because \text{ replacing } \psi \text{ by } s)$$

$$u(x,t) = u(x-bt,0) + \int_{0}^{t} f(x+b(s-t), s)ds$$

$$u(x,t) = g(x-bt) + \int_{0}^{t} f(x+b(x-t), s)ds \qquad (x \in R^n, t \geq 0)$$

It is the required solution of initial value problem for non-homogeneous transport equation.

2.4 Laplace's Equation and its Fundamental Solution

We get the Laplace's equation in several physical phenomenon such as irrotational flow of incompressible fluid, diffusion problem etc. Let $U \subset R^n$ be a open set, $x \in R$ and the unknown is $u : \bar{U} \to R$, $u = u(x)$ then, the Laplace's equation is defined as

$$\Delta u = 0 \qquad \dots (1)$$

and Poisson's equation

$$-\Delta u = f$$

where the function $f : U \to R$ is given.

and also remember that the Laplacian of u is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

Definition: Harmonic function

A C^2 function u satisfying the Laplace's equation $\Delta u = 0$ is called a harmonic function.

Theorem: Find the fundamental solution of the Laplace's equation (1).

Solution: Probably, it is to be noted that the Laplace equation is invariant under rotation. So we attempt to find a solution of Laplace's equation (1) in $U = R^n$, having the form (radial solution)

$$u(x) = v(r), \qquad \dots(2)$$

where $r = |x| = \left(x_1^2 + ... + x_n^2\right)^{1/2}$ and v is to be selected (if possible) so that $\Delta u = 0$ holds.

We note that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + ... + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \qquad (x \neq 0)$$

for i=1,2,…,n.

Thus, we have

$$u_{x_i} = v'(r)\frac{x_i}{r},$$

and $u_{x_i x_i} = v''(r)\dfrac{x_i^2}{r^2} + v'(r)\left(\dfrac{1}{r} - \dfrac{x_i^2}{r^3}\right)$

for i=1,…,n.

So

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} v''(r)\left(\frac{x_i^2}{r}\right) + v'(r)\left\{\sum_{i=1}^{n}\left(\frac{1}{r} - \frac{x_i^2}{r^2}\right)\right\} = v''(r) + \frac{n-1}{r}v'(r)$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0$$

If $v' \neq 0$, we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r},$$

Integrating w. r. t. r,

$$\log v' = -(n-1)\log r + \log a$$

where $\log a$ is a constant.

Now, $\qquad v' = \dfrac{a}{r^{n-1}}$

Again integrating

$$v(r) = \begin{cases} a\log r + b & (n = 2) \\ \dfrac{a}{r^{n-2}} + b & (n \geq 3) \end{cases}$$

where a and b are constants.

Therefore, if r >0, the solution of Laplace's equation is

$$u(x)=\begin{cases} a\log|x|+b & (n=2) \\ \dfrac{a}{|x|^{n-2}}+b & (n\ge 3) \end{cases}$$

Without loss of generality, we take b=0. To find b, we normalize the solution i.e.

$$\int_{R^n} u(x)dx=1$$

So, the solution is

$$u(x)=\begin{cases} -\dfrac{1}{2\pi}\log|x| & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n\ge 3) \end{cases} \qquad \ldots (3)$$

for each $x\in R^n, x\neq 0$ and $\alpha(n)$ is the volume of the unit ball in R^n.

We denote this solution by $\Phi(x)$ and

$$\Phi(x)=\begin{cases} \dfrac{-1}{2\pi}\log|x| & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n\ge 3) \end{cases} \qquad \ldots (4)$$

defined for $x\in R^n, x\neq 0$, is the fundamental solution of Laplace's equation.

Remarks: 1. We conclude that the solution of Laplace's equation $\Delta u=0$, $\Phi(x)$ is harmonic for $x\neq 0$. So the mapping $x\to\Phi(x)$, $x\neq 0$ is harmonic.

2. Shifting the origin to a new point y, the Laplace's equation remains unchanged. So $\Phi(x-y)$ is harmonic for $x\neq y$. If $f:R^n\to R$ is harmonic, then $\Phi(x-y)\,f(y)$ is harmonic for each $y\in R^n$ and $x\neq y$.

3. If we take the sum of all different points y over R^n, then

$$\int_{R^n}\Phi(x-y)\,f(y)\,dy \quad \text{is harmonic.}$$

Since $\Delta u(x)=\int_{R^n}\Delta_x\Phi(x-y)\,f(y)\,dy$

is not valid near the singularity $x=y$.

We must proceed more carefully in calculating Δu.

2.4.1 Fundamental Solution of Poission's Equation

To solve the Poission equation is $\Delta u = -f$, where $x \in U \subseteq R^n$, $f: R^n \to R$, U is an open set and unknown function is $u: \overline{U} \to R.$

Solution: We know that $x \to \Phi(x-y) f(y)$ for $x \neq y$ is harmonic for each point $y \in R^n$, and so is the sum of finitely many such expression constructed for different points y Consider the convolution

$$u(x) = \int_{R^n} \Phi(x-y) f(y) dy \qquad \ldots (5)$$

Form equations (4) and (5), we have

$$u(x) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\int_{R^n} \log(|x-y|) f(y) dy & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{R^n} \dfrac{f(y)}{|x-y|^{n-2}} dy & (n \geq 3) \end{cases} \qquad \ldots (6)$$

For simplicity, we assume that the function f used in Poission's equation is twice continuously differentiable. Now, we show that $u(x)$ defined by equation (5) satisfies

(i) $u \in C^2(R^n)$

(ii) $\Delta u = -f$ in R^n.

Consequently, the function in (6) provided us with a formula for a solution of Poission's equation.

Proof of (i):

Define u as,

$$u(x) = \int_{R^n} \Phi(x-y)\, f(y) dy$$

By change of variable $x - y = z$

$$u(x) = \int_{R^n} \Phi(x) f(x-z) dz = \int_{R^n} \Phi(x) f(x-y) dy$$

By definition . u_{x_i} .is

$$\frac{u(x+he_i)-u(x)}{h} = \int_{R^n} \Phi(y) \left[\frac{f(x+he_i) - f(x)}{h} \right] dy \qquad (*)$$

where $h \neq 0$ is a real number $e_i \in R^n$, $e_i = (0,0,...,0,1,0,...,0)$ with 1 in the ith position.

Thus, on taking $h \to 0$ in equation (*), we have

$$\frac{\partial u(x)}{\partial x_i} = \int_{R^n} \Phi(y)\left\{\frac{\partial f(x-y)}{\partial x_i}\right\}dy \qquad (**)$$

for $i = 1, 2, 3, ..., n$

Similarly,

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{R^n} \Phi(y)\left\{\frac{\partial^2 f(x-y)}{\partial x_i \partial x_j}\right\}dy \qquad (***)$$

for $i, j = 1, 2, ..., n$

As the expression on the right hand side of equation (***) is continuous in the variable x, we see that

$$u \in C^2(R^n)$$

This proves (i).

Proof of (ii)

(ii) From part (i), we have

$$\Delta u(x) = \int_{R^n} \Phi(y)\Delta_x f(x-y)\,dy$$

Since $\Phi(y)$ is singular at $y = 0$, so we include it in small ball $B(0,\varepsilon)$, where $\varepsilon > 0$

Then,

$$\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy + \int_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy$$

$$= I_\varepsilon + J_\varepsilon \qquad \ldots(7)$$

where

$$I_\varepsilon = \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy \qquad \ldots\ (8)$$

$$J_\varepsilon = \int_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy \qquad \ldots\ (9)$$

Now,

$$|I_\varepsilon| = \left|\int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy\right|$$

$$\le c\|D^2 f\|_{L^\infty(R^n)} \int_{B(0,\varepsilon)} |\Phi(y)|\,dy$$

$$\le \begin{cases} c\varepsilon^2 |\log\varepsilon| & (n=2) \\ c\varepsilon^2 & (n \ge 3) \end{cases}$$

Now,

$$J_\varepsilon = \int\limits_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy$$

$$= \int\limits_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\,dy \qquad \left(\because \frac{\partial}{\partial x} = -\frac{\partial}{\partial y}, \Delta_x = \Delta_y\right)$$

Integrating by parts

$$J_\varepsilon = -\int\limits_{R^n - B(0,\varepsilon)} D\Phi(y) D_y f(x-y)\,dy + \int\limits_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial \nu} ds(y)$$

where ν denoting the inward pointing unit normal along $\partial B(0,\varepsilon)$.

$$J_\varepsilon = K_\varepsilon + L_\varepsilon$$

We take,

$$|L_\varepsilon| = \left| \int\limits_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial \nu} ds(y) \right|$$

$$\le \|Df\|_{L^\infty(R^n)} \int\limits_{\partial B(0,\varepsilon)} |\Phi(y)|\,ds(y)$$

$$|L_\varepsilon| \le \begin{cases} c\varepsilon|\log \varepsilon| & (n=2) \\ c\varepsilon & (n \ge 3) \end{cases} \qquad \ldots (10)$$

Now $$K_\varepsilon = -\int\limits_{R^n - B(0,\varepsilon)} D\Phi(y) D_y f(x-y)\,dy$$

Integrating by parts

$$K_\varepsilon = \int\limits_{R^n - B(0,\varepsilon)} \Delta\Phi(y) f(x-y)\,dy - \int\limits_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial \nu} f(x-y)\,ds(y)$$

$$= -\int\limits_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial \nu} f(x-y)\,ds(y) \qquad \text{(since } \Phi \text{ is harmonic)}$$

$$D\Phi(y)=\left(\frac{\partial\Phi}{\partial y_1},\quad \frac{\partial\Phi}{\partial y_2},\ldots,\quad \frac{\partial\Phi}{\partial y_n}\right)$$

$$Also\quad \frac{\partial\Phi}{\partial y_i}=\frac{\partial}{\partial y_i}\left(\frac{1}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-2}}\right)$$

$$=\frac{-(n-2)}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-1}}\frac{\partial|y|}{\partial y_i}=\frac{-1}{n\alpha(n)|y|^{n-1}}\frac{y_i}{|y|}$$

$$=\frac{-1}{n\alpha(n)}\frac{y}{|y|^n}\qquad (y\neq 0)$$

and $\qquad \nu=\dfrac{-y}{|y|}=\dfrac{-y}{\varepsilon}\qquad$ on $\qquad \partial B(0,\varepsilon)$

So,

$$\frac{\partial\Phi(y)}{\partial\nu}=\nu.D\Phi(y)=\frac{1}{n\alpha(n)\varepsilon^{n-1}}\qquad \text{on } \partial B(0,\varepsilon)$$

Now, we have

$$K_\varepsilon=-\frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(0,\varepsilon)} f(x-y)\,ds(y)$$

$$K_\varepsilon=-\oint_{\partial B(x,\varepsilon)} f(y)\,ds(y)\to -f(y)\quad \text{as } \varepsilon\to 0 \qquad \ldots (11)$$

Combining equations (5) to (11) and letting $\varepsilon\to 0$, we have

$$\Delta u(x)=-f(x)$$

This completes the proof. Thus $u(x)$ given by (5) is the solution of Poission's equation.

2.4.2 Some Important Properties (in Polar coordinates)

(i) $\displaystyle\int_{R^n} f\,dx=\int_0^\infty\int_{\partial(x,r)}(f\,ds)\,dr$

(ii) $\displaystyle\int_{B(x_0,r)} f\,dx=\int_0^r\left(\int_{\partial B(x_0,r)} f\,ds\right)dr$

(iii) $\displaystyle\frac{d}{dr}\left(\int_{B(x_0,r)} f\,dx\right)=\int_{\partial B(x_0,r)} f\,ds$

2.5 Mean-value Theorem

Theorem: Mean-value formulas for Laplace's equation

If u is a harmonic function. Then

$$u(x) = \oint_{\partial B(x,r)} uds = \oint_{B(x,r)} udy \qquad \ldots (1)$$

for each ball $B(x,r) \subset U$.

OR

If u is harmonic function, prove that u equals to both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$ provided $B(x,r) \subset U$.

Proof: (Proof of Part I)

Set $$\Phi(r) := \oint_{\partial B(x,r)} u(y)\,ds(y) \qquad \textbf{... (2)}$$

Shifting the integral to unit ball, if z is an arbitrary point of unit ball then

$$\Phi(r) := \oint_{\partial B(x,r)} u(x+rz)\,ds(z)$$

Then

$$\Phi'(r) = \oint_{\partial B(0,1)} Du(x+rz).z\,ds(z)$$

And consequently, using Green's formulas, we have

$$\Phi'(r) = \oint_{\partial B(x,r)} Du(y).\frac{y-x}{r}\,ds(y)$$

$$= \oint_{\partial B(x,r)} Du(y).\nu\,ds(y)$$, where ν is unit outward normal to $\partial B(x,r)$.

$$\Phi'(r) = \oint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\,ds(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\,dy$$

$$= \frac{r}{n} \oint_{B(x,r)} \Delta u(y)\,dy = 0 \qquad (\because \Delta u = 0 \;\; on \;\; B(x,r))$$

Hence Φ is constant and

$$\Phi(r) = \lim_{t\to 0}\Phi(t) = \lim_{t\to 0} \oint_{\partial B(x,t)} u(y)\,ds(y) = u(x) \qquad \ldots(3)$$

From (2) and (3), we have

$$u(x) = \oint_{\partial B(x,r)} u(y)\,ds(y) \qquad \ldots\ (4)$$

(Proof of Part II)

Using coarea formula, we have

$$\int_{B(x,r)} u\,dy = \int_0^r \left(\int_{\partial B(x,t)} u\,ds \right) dt$$

$$= \int_0^r u(x)\, n\alpha(n)\, t^{n-1}\, dt$$

$$= u(x)\alpha(n) r^n$$

$$\Rightarrow u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u\,dy$$

$$= \oint_{B(x,r)} u\,dy \qquad \ldots\ (5)$$

From (4) and (5), we have

$$u(x) = \oint_{\partial B(x,r)} u\,ds = \oint_{B(x,r)} u\,dy$$

Hence proved.

Converse of Mean- value Theorem

Theorem: If $u \in C^2(U)$ satisfies the mean value formula

$$u(x) = \oint_{\partial B(x,r)} u\,ds$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof: Suppose that u is not harmonic, so $\Delta u \neq 0$. Therefore there exists a ball $B(x,r) \subset U$ such that $\Delta u > 0$ within $B(x,r)$.

But then for Φ, we know that

$$0 = \Phi'(r) = \frac{r}{n} \oint_{B(x,r)} \Delta u(y)\,dy > 0$$

which is a contradiction. Hence u is harmonic in U .

2.6 Properties of Harmonic Functions

Here, we present an interesting deduction about the harmonic function, all based upon the mean-value formula by assuming the following properties that $U \subset R^n$ is open and bounded.

2.6.1 Strong Maximum Principle, Uniqueness

Theorem: Let $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.

(i) Then $\max_{\bar{U}} u = \max_{\partial U} u$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u,$$

then u is constant within U.

Assertion (i) is the maximum principle for Laplace's equation and (ii) is the strong maximum principle.

Proof: (ii) Suppose there exist a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u = M \qquad \ldots (1)$$

Then for $0 < r < dist(x_0, \partial U)$, the mean value property implies

$$\begin{aligned} M &= u(x_0) = \oint_{B(x_0,r)} u dy \\ &= \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u dy \\ &\le \frac{M}{\alpha(n) r^n} \int_{B(x_0,r)} dy \\ &\le M \end{aligned}$$

Equality holds only if $u = M$ within $B(x_0, r)$. So we have, $u(y) = M$ for all $y \in B(x_0, r)$. To show that this result holds for the set U.

Consider the set

$$X = \{x \in U \mid u(x) = M\}$$

We prove that X is both open and closed.

X is closed since if x is the limit point of set X, then $\exists$ a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$

Since u is continuous so $\{u(x_n)\} \to u(x)$.

So $\quad u(x) = M$

$\Rightarrow x \in X$

$\Rightarrow X$ is closed.

To show that X is open, let $x \in X$, there exists a ball $B(x,r) \subset U$ such that

$$u(x) = \oint_{B(x,r)} u dy$$

So $x \in B(x,r) \subset X$.

Hence X is open.

But U is connected. The only set which is both open and closed in U is itself U. So $U = X$.

Hence $u(x) = M \quad \forall x \in U$. So u is constant in U.

(i) Using above result, we have $u(y) \le u(x_0)$ for some y and suppose $x_0 \notin \partial U$.

Since U is harmonic, so by mean value theorem, there exists a ball $B(x_0, r) \subset U$ such that

$$u(x_0) = \oint_{\partial B(x_0,r)} u ds(y)$$

$$M \le \frac{1}{n\alpha(n) r^{n-1}} |u(y)| \left| \int_{\partial B(x_0,r)} ds(y) \right|$$

$$\le |u(y)|$$

Maximum value is less than $|u(y)|$, which is a contradiction.

Hence $x_0 \in \partial U$.

Remarks: 1. If U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\Delta u = 0 \text{ in } U$$

$$u = g \text{ on } \partial U$$

where $g > 0$.

Then u is positive everywhere in U if g is positive somewhere on ∂U.

2. An important application of maximum modulus principle is establishing the uniqueness of solutions to certain boundary value problem for poission's equation.

Theorem: (Uniqueness)

Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem

$$-\Delta u = f \text{ in } U$$

$$u = g \text{ on } \partial U$$

Proof: Let u and $\bar{u}$ be two solutions of given boundary value problem, then

$$-\Delta u = f \text{ in } U$$

$$u = g \text{ on } \partial U$$

and

$$-\Delta \bar{u} = f \text{ in } U$$

$$\bar{u} = g \text{ on } \partial U$$

Let $w = \pm(u - \bar{u})$

$$\Delta w = 0 \text{ in } U$$

$$w = 0 \text{ on } \partial U$$

$\Rightarrow w$ is harmonic in U and w attains maximum value on boundary which is zero. If U is connected then w is constant. So $w = 0$ in U

Hence $u = \bar{u}$ in U .

2.6.2 Regularity

In this property, we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus harmonic functions are automatically infinitely differentiable.

Theorem: If $u \in C(U)$ satisfies the mean value property for each ball $B(x,r) \subset U$, then

$$u \in C^\infty(U)$$

Proof: Define a set $U_\varepsilon = \{x \in U \,|\, dist(x, \partial U) > \varepsilon\}$ and η be a standard mollifier.

Set $\quad u^\varepsilon = \eta_\varepsilon * u \quad \text{in} \quad U_\varepsilon \qquad \ldots (1)$

We first show that $u^\varepsilon \in C^\infty(U_\varepsilon)$.

Fix $x \in U_{\varepsilon}$, where $x = (x_1, x_2, ..., x_n)$.

Let h be very small such that $x + he_i \in U_{\varepsilon}$.

$$u^{\varepsilon}(x) = \eta_{\varepsilon} * u$$

$$= \frac{1}{\varepsilon^n} \int_{U_\varepsilon} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,dy \qquad \dots (2)$$

$$u^{\varepsilon}(x + he_i) = \frac{1}{\varepsilon^n} \int_{U_\varepsilon} \eta\left(\frac{x-y+he_i}{\varepsilon}\right) u(y)\,dy \qquad \dots (3)$$

Now using (2) and (3), we have

$$\frac{u^{\varepsilon}(x+he_i) - u^{\varepsilon}(x)}{h} = \frac{1}{\varepsilon^n} \int_{U_\varepsilon} \left[\frac{\eta\left(\frac{x-y+he_i}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)}{h}\right] u(y)\,dy$$

Taking the limit as $h \to 0$

$$\frac{\partial u^{\varepsilon}}{\partial x_i} = \frac{1}{\varepsilon^{n+1}} \int_{U_\varepsilon} \frac{\partial \eta\left(\frac{x-y}{\varepsilon}\right)}{\partial x_i} u(y)\,dy = \int_{U_\varepsilon} \frac{\partial \eta_\varepsilon(x-y)}{\partial x_i} u(y)\,dy$$

Since $\eta \in C^{\infty}(R^n)$. So $\frac{\partial u^{\varepsilon}}{\partial x_i}$ exists.

Similarly $D^{\alpha} u^{\varepsilon}$ exists for each multi-index α.

So $u^{\varepsilon} \in C^{\infty}(U_{\varepsilon})$.

We now show that $u = u^{\varepsilon}$ on U_{ε}.

Let $x \in U_{\varepsilon}$ then

$$u^{\varepsilon}(x) = \int_U \eta_{\varepsilon}(x-y) u(y)\,dy$$

$$= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$$= \frac{1}{\varepsilon^n} \int_0^{\varepsilon} \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x,r)} u(y)\, ds \right) dr \qquad \text{(using the cor. of coarea formula)}$$

$$= \frac{1}{\varepsilon^n} \int_0^{\varepsilon} \eta\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1} u(x)\, dr \qquad \text{(by Mean value formula)}$$

$$= \frac{n\alpha(n)u(x)}{\varepsilon^n} \int_0^{\varepsilon} \eta\left(\frac{r}{\varepsilon}\right) r^{n-1} dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_{B(0,\varepsilon)} \eta\left(\frac{y}{\varepsilon}\right) dy$$

$$= u(x) \int_{B(0,\varepsilon)} \eta_{\varepsilon}(y)\, dy \qquad \text{(by definition)}$$

$$= u(x)$$

So $u^{\varepsilon} \equiv u$ in U_{ε} and so $u \in C^{\infty}\left(U_{\varepsilon}\right)$ for each $\varepsilon > 0$.

Note: The above property holds for each $\varepsilon > 0$. It may happen u may not be smooth or even continuous upto ∂U .

2.6.3 Local Estimate for Harmonic Functions

Theorem: Suppose u is harmonic in U . Then

(i) $$\left|D^{\alpha}u\left(x_0\right)\right| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1\left(B\left(x_0,r\right)\right)} \qquad \ldots (1)$$

For each ball $B(x_0,r) \subset U$ and each multiindex α of order $|\alpha| = k$.

(ii) $$C_0 = \frac{1}{\alpha(n)} \quad , \quad C_k = \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)} \qquad (k = 1, \ldots) \qquad \ldots (2)$$

Proof: We prove this by induction.

For $k = 0, \alpha = 0$.

To show $\left|u\left(x_0\right)\right| \le \dfrac{1}{r^n \alpha(n)} \|u\|_{L^1}\left(B(x,r)\right)$

By mean value theorem

$$u\left(x_0\right) = \oint_{B\left(x_0,r\right)} u(y)\, dy \quad \text{for each ball } B\left(x_0,r\right) \subset U$$

$$u(x_0) = \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))} \qquad \ldots\ (3)$$

$$|D^0 u(x_0)| \le \frac{C_0}{r^n} \|u\|_{L^1(B(x_0,r))}$$

Hence the result.

For k=1, To show

$$|Du(x_0)| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

where $$C_1 = \frac{2^{n+1} n}{\alpha(n)}$$

Consider

$$\Delta u_{x_i} = \frac{\partial^2}{\partial x_1^2}(u_{x_i}) + \ldots + \frac{\partial^2}{\partial x_n^2}(u_{x_i})$$

$$= \frac{\partial}{\partial x_i}(\Delta u) = 0$$

So, u_{x_i} is harmonic. By mean value theorem

$$|u_{x_i}(x_0)| = \left| \oint_{B\left(x_0,\frac{r}{2}\right)} u_{x_i}\,dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0,\frac{r}{2}\right)} u_{x_i}\,dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0,\frac{r}{2}\right)} u v_i\,ds \right| \qquad \text{(By Gauss- Green Theorem)}$$

$$=\left|\frac{2^n}{r}\right|\left|\oint_{\partial B\left(x_0,\frac{r}{2}\right)} uv_i ds\right|$$

$$\leq \frac{2^n}{r}\|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)} \quad \ldots \text{ (4)}$$

If $x\in\partial B\left(x_0,\frac{r}{2}\right)$ then $B\left(x,\frac{r}{2}\right)\subset B(x_0,r)\subset U$.

By equation (3)

$$|u(x)|\leq\frac{2^n}{\alpha(n)r^n}\|u\|_{L^1\left(B\left(x,\frac{r}{2}\right)\right)}$$

$$\leq\frac{2^n}{\alpha(n)r^n}\|u\|_{L^1\left(B\left(x_0,r\right)\right)}$$

Hence

$$\|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)}\leq\frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n\|u\|_{L^1\left(B\left(x_0,r\right)\right)} \quad \ldots \text{ (5)}$$

From (4) and (5)

$$\left|u_{x_i}\left(x_0\right)\right|\leq\frac{2^{n+1}.n}{\alpha(n)r^{n+1}}\|u\|_{L^1\left(B\left(x_0,r\right)\right)}$$

$$\Rightarrow\left|D^\alpha u\left(x_0\right)\right|\leq\frac{C_1}{r^{n+1}}\|u\|_{L^1\left(B\left(x_0,r\right)\right)}$$

Hence result is true for k=1.

Assume that result is true for each multiindex of order less than or equal to k-1 for all balls in U . Fix $B(x_0,r)\subset U$ and α be multiindex with $|\alpha|=k$

$$D^\alpha u=\left(D^\beta u\right)_{x_i} \text{ for some } i=(1,2,3,\ldots,n)$$

where $|\beta|=k-1$. Consider the ball $B\left(x_0,\frac{r}{k}\right)$

$$\left|D^{\alpha}u(x_0)\right| = \left|\left(D^{\beta}u\right)_{x_i}\right|$$

$$\leq \frac{kn}{r}\left\|D^{\beta}u\right\|_{L^{\infty}\left(\partial B\left(x_0,\frac{r}{k}\right)\right)} \qquad \ldots (6)$$

If $x \in \partial B\left(x_0, \frac{r}{k}\right)$ then

$$B\left(x, \frac{k-1}{k}r\right) \subset B(x_0, r) \subset U$$

By assumption, in the ball $B\left(x, \frac{k-1}{k}r\right)$

$$\left|D^{\beta}u(x_0)\right| \leq \frac{\left[2^{n+1}n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}}\|u\|_{L^1\left(B\left(x,\frac{k-1}{k}r\right)\right)} \qquad \ldots (7)$$

From (6) and (7)

$$\left|D^{\alpha}u(x_0)\right| \leq \frac{kn}{r}\frac{\left[2^{n+1}n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}}\|u\|_{L^1(B(x_0,r))}$$

$$\leq \frac{\left(2^{n+1}nk\right)^{k}}{\alpha(n)r^{n+k}}\|u\|_{L^1(B(x_0,r))}$$

Since,

$$\frac{1}{2}\left[\frac{k}{2(k-1)}\right]^{n} < 1 \quad \text{for all} \quad k \geq 2$$

Hence result holds for $|\alpha| = k$.

2.6.4 Liouville's Theorem

We see that there are no nontrivial bounded harmonic functions on all of R^n

Theorem: Suppose $u : R^n \to R$ is harmonic and bounded. Then u is constant.

Proof: Let $x_0 \in R^n, r > 0$, then by mean value theorem

$$\left|Du(x_0)\right| = \left|u_{x_i}(x_0)\right| = \left| \oint_{B\left(x_0,\frac{r}{2}\right)} u_{x_i} dx \right|$$

$$= \left| \frac{2^n}{\alpha(n) r^n} \int_{\partial B\left(x_0,\frac{r}{2}\right)} uvds \right| \qquad \text{(By Guass Green's theorem)}$$

$$\leq \frac{2^n}{r} \|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)}$$

If $x \in \partial B\left(x_0, \frac{r}{2}\right)$ then $B\left(x, \frac{r}{2}\right) \subset B(x_0, r)$

$$|u(x)| \leq \frac{1}{\alpha(n)} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))}$$

Hence

$$\left|u_{x_i}(x_0)\right| \leq \frac{2n}{r}\left(\frac{2}{r}\right)^n \frac{1}{\alpha(n)} \|u\|_{L^1(B(x_0,r))}$$

$$= \frac{2^{n+1} n}{r^{n+1}\alpha(n)} \|u\|_{L^1(B(x_0,r))}$$

$$\leq \frac{n 2^{n+1}}{r} \|u\|_{L^\infty(R^n)} \to 0 \text{ as } r \to 0$$

Hence $Du = 0$.

So u is constant.

Theorem: Representation Formula

Let $f \in C_c^2(R^n), n \geq 3$. Then any bounded solution of $-\Delta u = f$ in R^n (1)

of the form

$$u(x) = \int_{R^n} \Phi(x-y) f(y) dy + c \qquad (x \in R^n)$$

For some constant c and $\Phi(x)$ is the solution of Laplace's equation.

Proof: Since $\Phi(x) \to 0$ as $|x| \to \infty$ for $n \geq 3$

$\Rightarrow \Phi(x)$ is bounded.

Let u be a solution of equation (1) which is represented as

$$u = \int_{R^n} \Phi(x-y)\, f(y)\, dy$$

and it is bounded.

Since $f \in C^2(R^n)$ and $\Phi(x)$ is bounded for $n \geq 3$. Let $\bar{u}$ be another bounded solution of equation (1)

Define $w = u - \bar{u}$

$$\Delta w = 0$$

and w is bounded ($\because$ difference of two bounded functions)

By Liouville's theorem

$$w = \text{constant}$$

$$\text{or } u - \bar{u} = -c$$

$$\Rightarrow \bar{u} = u + c$$

This is the required result.

Note: For n=2, $\Phi(x) = \frac{-1}{2\pi} \log|x|$ is unbounded as $|x| \to \infty$ and so may be

$$\int_{R^2} \Phi(x-y)\, f(y)\, dy$$

2.6.5 Analytically

Theorem: If u is harmonic in U then u is analytic in U .

Proof: Suppose that x_0 be any point in U . Firstly, we show that u can be represented by a convergent power series in some neighbourhood of x_0 .

Let $\mu = \frac{1}{4} dist(x_0, \partial U)$

Then $M = \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,2r))} < \infty$ … (1)

for each $x \in B(x_0, r), B(x, r) \subset B(x_0, 2r) \subset U$

By estimates of derivatives

$$\left|D^\alpha u(x_0)\right| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

where $c_k = \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)}$ for each $|\alpha| = k$

So $$\left\|D^\alpha u(x_0)\right\|_{L^\infty(B(x_0,r))} \le \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)r^{n+k}}\|u\|_{L^1(B(x_0,r))}$$

$$\le M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|} \quad \ldots (2)$$

By Sterling formula

$$\lim_{k\to 0}\frac{k^{k+\frac{1}{2}}}{k!e^k} = \frac{1}{\sqrt{2\pi}}$$

$\Rightarrow k^k \le ck!e^k$, where c is constant.

Hence,

$$|\alpha|^{|\alpha|} \le ce^{|\alpha|}|\alpha|! \quad \ldots (3)$$

for some constant c and all multi indices α.

Furthermore, the Multinomial theorem implies

$$n^k = (1+\ldots+1)^k = \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!} \quad \ldots (4)$$

where $|\alpha|! \le n^{|\alpha|}\alpha!$

Using (4) and (3) in (2)

$$\left\|D^\alpha u(x_0)\right\|_{L^\infty(B(x_0,r))} \le M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|} ce^{|\alpha|}n^{|\alpha|}\alpha!$$

$$\le Mc\left(\frac{2^{n+1}n^2e}{r}\right)^{|\alpha|}\alpha!$$

Taylor series for u at x_0 is

$$\sum_\alpha \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha$$

The sum taken over all multiindices.

We claim that this power series converges, provided

$$|x-x_0|<\frac{r}{2^{n+2}n^3e}$$

To verify this, let us compute for each N

The remainder term is

$$R_N(x)=\sum_{|\alpha|=N}^{\infty}\frac{D^\alpha u\left(x_0+t(x-x_0)\right)(x-x_0)^\alpha}{\alpha!}$$

For some $0\le t\le 1$, t depending on x.

$$\left|R_N(x)\right|\le cM\sum\left(\frac{2^{n+1}n^2e}{r}\right)^N\left(\frac{r}{2^{n+2}n^3e}\right)^N$$

$$\le cM\sum_{|\alpha|=N}\left(\frac{1}{2n}\right)^N$$

$$\le\frac{cM}{2^N}\to 0 \text{ as } N\to 0$$

$\Rightarrow$Series is converges.

So $u(x)$ is analytic in neighbourhood of x_0.

But x_0 is arbitrary point of U .

So u is analytic in U .

2.6.6 Harnack's Inequality

This inequality shows that the values of non-negative harmonic functions within open connected subset of U , are comparable.

Theorem: For each connected open set $V\subset U$, $\exists$ a positive constant c, depending only on V , such that

$$\sup_V u\le c\inf_V u \qquad \ldots \quad (1)$$

For all nonnegative harmonic functions u in U .

Thus in particular

$$\frac{1}{c}u(y)\le u(x)\le cu(y) \qquad \forall x,y\in V$$

Proof: Let $r=\frac{1}{4}dist(V,\partial U)$

Choose $x, y \in V, |x-y| \leq r$. Then

$$u(x) = \oint_{B(x,2r)} udz \geq \frac{1}{\alpha(n) 2^n r^n} \int_{B(y,r)} udz$$

$$= \frac{1}{2^n} \oint_{B(y,r)} udz = \frac{1}{2^n} u(y)$$

$$\Rightarrow 2^n u(x) \geq u(y) \qquad \ldots (2)$$

Interchanging the role of x and y

$$2^n u(y) \geq u(x) \qquad \ldots (3)$$

Combining (2) and (3)

$$2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y) \qquad x, y \in V$$

Since V is connected, $\bar{V}$ is compact, so $\bar{V}$ can be covered by a chain of finite number of balls $\{B_i\}_{i=1}$ such that $B_i \cap B_j \neq 0$ for $i \neq j$ each of radius $\frac{r}{2}$.

Therefore,

$$u(x) \geq \frac{1}{2^{nN}} u(y) \qquad \forall x, y \in V$$

$$u(x) \geq \frac{1}{c} u(y)$$

Similarly,

$$cu(y) \geq u(x)$$

So,
$$\frac{1}{c} u(y) \leq u(x) \leq cu(y) \qquad \forall x, y \in V$$

2.7 Green's Function:

Suppose that $U \subset R^n$ is open, bounded and ∂U is C^1. We introduced general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U \qquad \ldots (1)$$

subjected to the prescribed boundary condition

$$u = g \quad \text{on } \partial U \qquad \ldots (2)$$

Theorem: (Derivative of Green's function)

Derive the Green's function of equation (1) under the initial condition (2).

Proof: Let $u \in C^2\left(\bar{U}\right)$ is an arbitrary function and fix $x \in U$, choose $\varepsilon > 0$ so small that $B(x,\varepsilon) \subset U$ and apply Green's formula on the region $V_\varepsilon = U - B(x,\varepsilon)$ to $u(y)$ and $\Phi(y-x)$.

Then, we have

$$\int_{V_\varepsilon} \left[u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\right]dy$$

$$= \int_{\partial V_\varepsilon} \left[u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u(y)}{\partial \nu}\right]ds(y)$$

where ν denoting the outer unit normal vector on ∂V_ε. Also $\Delta\Phi(x-y) = 0$ for $x \neq y$.

Then

$$-\int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\,dy$$

$$= \int_{\partial U + \partial B(x,\varepsilon)} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu}\right]ds(y) \qquad \ldots (3)$$

Now

$$\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u(y)}{\partial \nu}ds(y)\right| \leq \|Du\|_{L^\infty(\partial B(x,\varepsilon))} \left|\Phi(y-x)\right|\left|\int ds(y)\right|$$

$$\leq c\frac{1}{\varepsilon^{n-2}} n\alpha(n)\varepsilon^{n-1} = o(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0 \qquad \ldots (4)$$

Also

$$\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi(y-x)}{\partial \nu}ds(y) = \int_{\partial B(0,\varepsilon)} u(y+x)\frac{\partial \Phi(y)}{\partial \nu}ds(y)$$

Now

$$D\Phi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} \qquad , y \neq 0$$

$$\nu = -\frac{y}{|y|} = \frac{-y}{\varepsilon} \quad \int_{\partial B(0,\varepsilon)} u(y+x)\frac{\partial \Phi(y)}{\partial \nu}ds(y) = \int_{\partial B(0,\varepsilon)} u(y+x)\frac{1}{n\alpha(n)\varepsilon^{n-1}}ds(y)$$

$$= \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y)\,ds(y)$$

$$= \oint_{\partial B(x,\varepsilon)} u(y)\,ds(y) \to u(x) \quad \text{as } \varepsilon \to 0 \qquad \ldots (5)$$

Using (4) and (5) in equation (3) and making $\varepsilon \to 0$

$$-\int_U \Phi(y-x)\Delta y dy = \int_{\partial U}\left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}\right]ds(y) + u(x)$$

$(y \neq x)$

Thus

$$u(x) = \int_{\partial U}\left[\Phi(y-x)\frac{\partial u}{\partial \nu} - u(y)\frac{\partial \Phi(y-x)}{\partial \nu}\right]ds(y) - \int_U \Phi(y-x)\Delta u(y)\,dy \qquad \ldots(6)$$

This identity is valid for any point $x \in U$ and for any function $u \in C^2(\overline{U})$ and it gives the solution of problem defined by equation (1) and (2) provided that $u(y), \frac{\partial u}{\partial \nu}$ are known on the boundary ∂U and the value of Δu in U. But $\frac{\partial u}{\partial \nu}$ is unknown to us along the boundary. Therefore, we have to eliminate $\frac{\partial u}{\partial \nu}$ to find the solution. For it, we define a correction term formula $\phi = \phi^x(y)$ (for fixed x) given by the solution of

$$\Delta \phi^x = 0 \quad \text{in U}$$

$$\phi^x = \Phi(y-x) \quad \text{on } \partial U \qquad \ldots. (7)$$

Let us apply Green's formula once more,

$$\int_U \left[u(y)\Delta\phi^x - \phi^x \Delta u(y)\right]dy = \int_{\partial U}\left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x\frac{\partial u}{\partial \nu}\right]ds$$

Then we have

$$-\int_U \phi^x \Delta u(y)\,dy = \int_{\partial U}\left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x\frac{\partial u}{\partial \nu}\right]dx \qquad \ldots (8)$$

Adding equation (6) and (8)

$$u(x) = -\int_U \left[\Phi(y-x) - \phi^x(y)\right]\Delta u(y)\,dy - \int_{\partial U}\frac{\partial\left[\Phi(y-x) - \phi^x(y)\right]}{\partial \nu}u(y)\,dy \qquad \ldots (9)$$

Now we define Green's function for the region U as

$$G(x,y) = \Phi(y-x) - \phi^x(y) \qquad (x, y \in U, x \neq y) \qquad \ldots (10)$$

From equation (9) and (10)

$$u(x) = -\int_U G(x,y)\Delta u(y)\,dy - \int_{\partial U} u(y)\frac{\partial G(x,y)}{\partial \nu}\,ds(x) \qquad \ldots (11)$$

where $\dfrac{\partial G(x,y)}{\partial \nu} = D_y G(x,y)$. $\nu(y)$ is the outer normal derivative of G with respect to the variable y. Also we observe that equation (11) is independent of $\dfrac{\partial u}{\partial \nu}$.

Hence the boundary value problem given by equation (1) and (2) can be solved in term of Green's function and solution is given by equation (11) is known as **Representation formula** for Green's

Function.

Note: Fix $x \in U$. Then regarding G as a function of y, we may symbolically write

$$-\Delta G = \delta_x \text{ in } U$$

$$G = 0 \text{ on } \partial U$$

where δ_x denoting the Dirac Delta function.

2.7.1 Symmetry of Green's Function

Theorem: Show that for all $x, y \in U, x \neq y,$ $G(x,y)$ is symmetric i.e. $G(x,y) = G(y,x)$.

Proof: For fix $x, y \in U, x \neq y$

Write

$$v(z) = G(x,z), w(z) = G(y,z) \qquad (z \in U)$$

Then

$$\Delta v(z) = 0(z \neq x), \Delta w(z) = 0(z \neq y)$$

and $w = v = 0$ on ∂U .

Applying Green's formula on $V = U - [B(x,\varepsilon) \cup B(y,\varepsilon)]$ for sufficiently small $\varepsilon > 0$ yields.

$$\int_{\partial B(y,\varepsilon)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) ds(z) = \int_{\partial B(x,\varepsilon)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) ds(y) \qquad \ldots (1)$$

ν denoting the inward pointing unit vector field on $\partial B(x,\varepsilon) \cup \partial B(y,\varepsilon)$.

Now w is smooth near x, so

$$\left| \int_{\partial B(x,\varepsilon)} \frac{\partial w}{\partial \nu} v ds \right| \le \|Dw\|_{\partial B(x,\varepsilon)} \left| \int ds \right|$$

$$\le c\varepsilon^{n-1} \to 0 \text{ as } \varepsilon \to 0 \qquad \dots \quad (2)$$

We know that $v(z) = \Phi(z-x) - \phi^x(z)$, where ϕ^x is smooth in U .Thus

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w ds = \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(x-z) w(z) ds = w(x)$$

Now we have

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \left(w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right) ds(z) \to w(x)$$

Similarly,

$$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} \left(v \frac{\partial w}{\partial \nu} - w \frac{\partial v}{\partial \nu} \right) ds(z) \to v(y)$$

Therefore from equation (1) , we have

$$w(x) = v(y)$$

$$\Rightarrow G(x,y) = G(y,x)$$

Hence proved.

2.7.2 Green's Function for a Half Space

Definition: If $x = (x_1, ..., x_{n-1}, x_n) \in R_+^n$, its reflection in the plane ∂R_+^n is the point

$$\tilde{x} = (x_1, ..., x_{n-1}, x_n).$$

Definition: Green's function for the half space R_+^n is

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) \qquad (x, y \in R_+^n, x \ne y)$$

Example: Solve the boundary value problem

$$\Delta u = 0 \text{ in } R_+^n$$

$$u = g \text{ on } \partial R_+^n$$

with the help of Green's function.

Solution: Let $x, y \in R_+^n, x \neq y$.

By definition, $G(x,y) = \Phi(y-x) - \phi^x(y)$

We choose the corrector term

$$\phi^x(y) = \Phi(y-\tilde{x}) \qquad \ldots (1)$$

where $\tilde{x}$ is reflection of x w. r. t. ∂R_+^n.

Clearly $\Delta\phi^x = 0$ in R_+^n

Now

$$\Phi(y-\tilde{x}) = \frac{1}{n(n-2)\alpha(n)|y-\tilde{x}|^{n-2}}, \qquad n \geq 3$$

$$\frac{\partial\Phi}{\partial y_1}(y-\tilde{x}) = -\frac{y_1 - x_1}{n\alpha(n)|y-\tilde{x}|^n}$$

$$\frac{\partial^2\Phi}{\partial y_1^2} = -\frac{1}{n\alpha(n)|y-\tilde{x}|^n} + \frac{(y_1-x_1)^2}{\alpha(n)|y-\tilde{x}|^{n+2}}$$

$$\frac{\partial^2\Phi}{\partial y_n^2} = -\frac{1}{n\alpha(n)|y-\tilde{x}|^n} + (y_n + x_n)^2$$

Adding $\Delta\Phi(y-\tilde{x}) = 0$ on ∂R_+^n $|y-x| = (y-\tilde{x})$

So $\Phi(y-\tilde{x}) = \Phi(y-x)$

Hence both conditions are satisfied.

So, Green's function

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) \text{ is well defined.}$$

So, using the representation formula

$$u(x) = 0 - \int_{\partial R_+^n} g(y)\frac{\partial G}{\partial \nu}(x,y)\,ds(y)$$

$$\frac{\partial G}{\partial \nu}(x,y) = DG.\hat{\nu} = -\frac{\partial G}{\partial y_n}(x,y)$$

$$\frac{\partial G}{\partial y_n} = \frac{\partial \Phi}{\partial y_n}(y-x) - \frac{\partial \Phi}{\partial y_n}(y-\tilde{x})$$

$$= -\left[\frac{y_n - x_n}{n\alpha(n)|y-x|^n} - \frac{y_n + x_n}{n\alpha(n)|y-x|^n}\right]$$

$$= \frac{2x_n}{n\alpha(n)|x-y|^n} \qquad \left(on\, \partial R_+^n, |y-x| = |y-\tilde{x}|\right)$$

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial R_+^n} \frac{g(y)}{|x-y|^n} ds(y) \qquad \left(x \in R_+^n\right)$$

This is the required solution and is known as Poisson’s formula.

The function

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \qquad \left(x \in R_+^n, y \in \partial R_+^n\right)$$

is Poisson’s kernel for R_+^n.

2.7.3 Green’s Function for a Ball

Definition: If $x \in R^n - \{0\}$, the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$

Definition: Green’s function for the unit ball is $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$ $\left(x, y \in B(0,1), x \neq y\right)$.

Example: Solve the boundary value problem

$$\Delta u = 0 \text{ in } B(0,1)$$

$$u = g \text{ on } \partial B(0,1)$$

with the help of Green’s function.

Solution: Fix any point $x \in B^0(0,1)$ and $y \neq x$

The Green’s function is given by

$$G(x,y) = \Phi(y-x) - \Phi(y)$$

We choose $\phi^x(y) = \Phi(|x|(y-\tilde{x}))$

where $\tilde{x}$ dual of x w. r. t. $\partial B(0,1)$

As we know $\Phi(y-x)$ is harmonic. So $\Phi(y-\tilde{x})$ is also harmonic for $y \neq x$. Similarly $|x|^{2-n}\,\Phi(y-\tilde{x})$ is harmonic for $y \neq x$.

Or $\Phi(|x|(y-\tilde{x}))$ is harmonic for $y \neq x$

So, $\Delta\phi^x = 0$ in $B(0,1)$

On $\partial B(0,1)$:

$$\phi(x) = \Phi(|x|(y-\tilde{x}))$$

But

$$|x|^2|y-\tilde{x}|^2 = |x|^2\left\{\left(y_1 - \frac{x_1}{|x|^2}\right)^2 + \ldots + \left(y_n - \frac{x_n}{|x|^n}\right)^2\right\}$$

$$= |x|^2\left\{|y|^2 + \frac{1}{|x|^2} - \frac{2xy}{|x|^2}\right\}$$

$$= \left|x^2 + 1 - 2xy\right| \qquad (\because |y| = 1)$$

$$= |x|^2 + |y|^2 - 2xy$$

$$= |x-y|^2$$

So $\quad \phi(x) = \Phi(|x|(y-\tilde{x})) = \Phi(y-x)$.

Hence both conditions of $\phi^x(y)$ are satisfied.

So

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \quad \text{is well defined.}$$

Hence solution of given problem is given by

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G}{\partial \nu}\,ds(y)$$

Now on $\partial B(0,1)$

$$\frac{\partial G}{\partial \nu} = \frac{\partial G}{\partial y}.\nu\,, \ \nu \text{ being the unit normal.}$$

$$= \frac{\partial G}{\partial y}\frac{y}{|y|} = \sum \frac{\partial G}{\partial y_i} y_i \qquad (\because |y| = 1)$$

$$\frac{\partial G}{\partial y_i} = \frac{x_i - y_i}{n\alpha(n)|x-y|^n} + \frac{y_i|x|^2 - x_i}{n\alpha(n)|x-y|^n}$$

$$= \frac{y_i|x|^2 - y_i}{n\alpha(n)|x-y|^n}$$

$$\frac{\partial G}{\partial \nu} = -\frac{\left(1-|x|^2\right)}{n\alpha(n)|x-y|^n}$$

Therefore we have

$$u(x) = \int_{\partial B(0,1)} g(y)\frac{1-|x|^2}{n\alpha(n)|x-y|^n} ds(y)$$

This is the required solution.

2.7.4 Energy Methods

Theorem: (Uniqueness)

There exists at most one solution $u \in C^2(\bar{U})$ of the boundary value problem

$$-\Delta u = f \quad \text{in } U$$

$$u = g \quad \text{on } \partial U$$

where U is open, bounded and ∂U is C^1.

Proof: Let $\bar{u}$ be another solution of given problem.

Let $w = u - \bar{u}$ then $\Delta w = 0$ in U

$$w = 0 \text{ on } \partial U$$

Consider

$$\int_U w\Delta w dx = \int_U w\left(w_{x_i}\right)_{x_i} dx$$

Integrating by parts

$$= -\int_U w_{x_i} w_{x_i} dx + \int_{\partial U} w_{x_i} w \nu ds \ , \ \nu \text{ being the unit normal}$$

$$= -\int_U |Dw|^2 dx + 0$$

$$\Rightarrow |Dw|^2 = 0 \quad \text{in} \quad U$$

$$\Rightarrow Dw = 0 \quad \text{in} \quad \text{U}$$

$$\Rightarrow w = \text{constant in } U$$

But $w = 0$ in ∂U

Hence $w = 0$ in U

$\Rightarrow u = \bar{u}$

Hence uniqueness of solution.

Dirichlet's Principle: Let us demonstrate that a solution of the boundary value problem for Poisson's equation can be characterized as the minimize of an appropriate functional.

Thus, we define the energy functional

$$I[w] = \int_U \frac{1}{2}|Dw|^2 - wfdx$$

w belonging to the admissible set $A = \left\{ w \in C^2(\bar{U}) \mid w = g \ on \partial U \right\}$

Theorem: Let $u \in C^2\left(\bar{U}\right)$ be a solution of Poisson's equation. Then

$$I[u] = \min_{w \in A} I[w] \qquad \ldots (1)$$

Conversely, if $u \in A$ satisfies (1) then u is a solution of boundary value problem

$$-\Delta u = f \quad \text{in } U$$

$$u = g \quad \text{on } \partial U \qquad \ldots (2)$$

Proof: Let $w \in A$ and u be a solution of Poisson's equation. So

$$-\Delta u = f \quad \text{in} \quad U$$

$$\Rightarrow 0 = \int_U (-\Delta u - f)(u - w)\,dx$$

$$= -\int_U \Delta u(u - w)\,dx - \int_U f(u - w)\,dx$$

Integrating by parts

$$0 = \int_U Du.D(u - w)\,dx - \int_{\partial U} (u - w) Du.vds - \int_U f(u - w)\,dx$$

$$= \int_U (Du.Du - fu)\,dx - 0 - \int_U (Du.Dw - fw)\,dx$$

$$\Rightarrow \int_U \left(|Du|^2 - fu \right) dx = \int_U (Du.Dw - fw)\, dx$$

$$\Rightarrow \int_U \left(|Du|^2 - fu \right) dx \le \int \left[\frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2 - fw \right] dx \qquad \text{(By Cauchy-Schwartz's inequality)}$$

So $\int \left[\frac{1}{2}|Du|^2 - fu \right] dx \le \int \left[\frac{1}{2}|Dw|^2 - fw \right] dx$

$$I[u] \le I[w]$$

Since $u \in A$, So

$$I[u] = \min_{w \in A} I[w]$$

Conversely, suppose that $I[u] = \min_{w \in A} I[w]$

For any $v \in C_c^\infty(U)$, define $i(\tau) = I[u + \tau v]$, $\tau \in R$

So $i(\tau)$ attains minimum for $\tau = 0$

$$i'(\tau) = 0 \quad \text{for} \quad \tau = 0$$

$$i(\tau) = \int_U \left[\frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v) f \right] dx$$

$$= \int_U \left[\frac{1}{2}\left(|Du|^2 + \tau^2 |Dv|^2 \right) + \tau DuDv - (u + \tau v) f \right] dx$$

$$i'(0) = \int_U [Du.Dv - vf] dx$$

Integration by parts

$$0 = -\int_U v\Delta u dx + \int_{\partial U} Du.\nu ds - \int v f dx$$

$$0 = \int_U [-\Delta u - f] v dx \qquad \left[\because v \in C_c^\infty(U) \right]$$

This is true for each function $v \in C_c^\infty(U)$.

So $\Delta u = -f$ in U .

So u is a solution of Poisson's equation.